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# $m$-Clouds in generalized hexagons

Leen Brouns*, Joseph A. Thas, Hendrik Van Maldeghem[1]

*Department of Pure Mathematics and Computeralgebra, University of Ghent, Galglaan 2, 9000 Ghent, Belgium*

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**Abstract**

In this paper, we define $m$-clouds in finite generalized hexagons and look for possible sizes of these pointsets. We also give some remarks on $m$-clouds and dense clouds in generalized quadrangles.


*Keywords:* Generalized hexagon; Generalized quadrangle; Projective plane; $m$-Ovoids

## 1. Introduction

In [7], Thas studied interesting pointsets in generalized quadrangles (e.g. $m$-ovoids), obtaining strongly regular graphs. By modifying the definition of $m$-ovoid, we can apply it to the case of the hexagons. The thus defined $m$-clouds are used to characterize thin subhexagons of a generalized hexagon (these are important in connection with regularity conditions and for characterizations of the classical hexagons). We are also able to extend 'small' $m$-clouds of any generalized hexagon to larger structures.

## 2. Definitions

A *generalized n-gon* $\Gamma = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *of order* $(s, t)$ is an incidence structure of points and lines with $s+1$ points incident with a line and $t+1$ lines incident with a point,

* Corresponding author.

*E-mail addresses:* lb@cage.rug.ac.be (L. Brouns), jat@cage.rug.ac.be (J.A. Thas), hvm@cage.rug.ac.be (H.V. Maldeghem).

[1] The third author is a Research Director of the Fund for Scientific Research—Flanders (Belgium).


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$s,t \geqslant 1$, such that $\Gamma$ has no ordinary $k$-gons for any $2 \leqslant k < n$, and where any two elements belong to some ordinary $n$-gon.

Distance between two elements $x, y$ is measured in the incidence graph, and denoted by $\delta(x,y)$. The set of elements at distance $i$ of an element $x$ is denoted by $\Gamma_i(x)$. If two points $x, y$ are at distance 2, we call them *collinear* and write $x \sim y$. If two points $x, y$ are at distance 4 and $n > 4$, then the unique point in $\Gamma_2(x) \cap \Gamma_2(y)$ is denoted by $x \bowtie y$. If two elements $x, y$ are at distance $k < n$, the projection of $x$ onto $y$ is the unique element of $\Gamma_{k-1}(x) \cap \Gamma_1(y)$ and is denoted by $\mathrm{proj}_y x$. If two elements $x, y$ are at maximal distance $n$, they are said to be opposite. For a survey on generalized polygons, see [6, Chapter 6; 8].

An *m-cloud* of $\Gamma$, $2 \leqslant m \leqslant t$, is a subset $\mathscr{C}$ of points of $\Gamma$ at mutual distance 4, such that $\forall x, y \in \mathscr{C} : x \bowtie y$ is collinear with exactly $m+1$ points of $\mathscr{C}$.

We put $\mathscr{C}^* = \{x \bowtie y \mid x, y \in \mathscr{C}\}$, throughout.

## 3. *m*-Clouds in generalized hexagons

**Lemma 1.** *Let $\Gamma$ be a generalized hexagon, and $\mathscr{C}$ an m-cloud of $\Gamma$. Then the points of $\mathscr{C}$ are collinear with a constant number $f+1$ of points in $\mathscr{C}^*$.*

**Proof.** Take a point $x \in \mathscr{C}$, and suppose $x$ is collinear with $f+1$ points $z_i \in \mathscr{C}^*$. For each $z_i$ there are $m$ points $y_{ij}$ in $\mathscr{C}$ collinear with $z_i$, and different from $x$. As $y_{ij} \neq y_{kl}$ if $i \neq k$ (otherwise there arises a quadrangle with vertex set $\{x, z_i, y_{ij} = y_{kl}, z_k\}$), $\mathscr{C}$ has at least $1 + (f+l)m$ points. As all points in $\mathscr{C}$ are at mutual distance 4, we counted all points in $\mathscr{C}$, hence $|\mathscr{C}| = 1 + (f+l)m$, and $f+1$ turns out to be a constant. $\square$

**Remark.** The geometry $\Gamma' = (\mathscr{C}, \mathscr{C}^*, \sim)$ clearly is a $2-(1+(f+l)m, m+1, 1)$-design. Hence the number of points in $\mathscr{C}^*$ is $(1+(f+1)m)(f+1)/(m+1)$. This last expression implies $(m+1) \mid f(f+1)$.

The parameter $f$ is called the *index* of the $m$-cloud. For $m$ and $f$ maximal (i.e. $f = m = t$), we know that $|\mathscr{C}| = |\mathscr{C}^*| = t^2 + t + 1$. For $f = t$, $m = t-1$, we have $|\mathscr{C}| = t^2$, $|\mathscr{C}^*| = t^2 + t$. (The values $f = t-1$, $m = t$ do not occur by the divisibility condition mentioned above.) We will consider in detail these two cases.

**Lemma 2.** *No two distinct points of $\mathscr{C}^*$ are collinear.*

**Proof.** Let $z$, $u$ be in $\mathscr{C}^*$ and suppose $\delta(z,u) = 2$. Take points $z'$ and $u'$ of $\mathscr{C}$ at distance 2 of $z$ and $u$, respectively. If $z' = u'$ then $\delta(z,u) = 4$, a contradiction with $\delta(z,u) = 2$. If $z' \neq u'$, then $\delta(z', u') = 4$ by definition of $\mathscr{C}$, hence there arises a $k$-gon, with $k < 6$. $\square$

**Theorem 3.** *If $\mathscr{C}$ is an m-cloud of index m, then the geometry $\Gamma' = (\mathscr{C}, \mathscr{C}^*, \sim)$ is a projective plane of order m. Hence $\mathscr{C}^*$ is also an m-cloud of index m, with $(\mathscr{C}^*)^* = \mathscr{C}$.*

**Proof.** As $\Gamma'$ is a $2-(m^2+m+1, m+1, 1)$-design, it is a projective plane of order $m$. By the duality principle in projective planes, $\mathscr{C}^*$ will also be an $m$-cloud of index $m$. □

**Theorem 4.** *If $\mathscr{C}$ is an $(f-1)$-cloud of index $f$, then the geometry $\Gamma' = (\mathscr{C}, \mathscr{C}^*, \sim)$ is an affine plane of order $f$.*

**Proof.** As $\Gamma'$ is a $2-(f^2, f, 1)$-design, this follows again from design theory. □

**Corollary 5.** *If $\mathscr{C}$ is an $m$-cloud with $|\mathscr{C}| \geqslant t^2+1$, then $\mathscr{C}$ is a $t$-cloud of index $t$, so $|\mathscr{C}| = t^2+t+1$. The geometry $\Gamma' = (\mathscr{C}, \mathscr{C}^*, \sim)$ is a projective plane of order $t$. The union $\mathscr{C} \cup \mathscr{C}^*$ is the point set of a thin ideal subhexagon of $\Gamma$ (i.e. a subhexagon with 2 points on a line and $t+1$ lines through a point).*

**Corollary 6.** *If $|\mathscr{C}| \geqslant t^2 - t + 2$, then either $|\mathscr{C}| = t^2$ or $t^2+t+1$.*
*If $|\mathscr{C}| = t^2$, then $\Gamma' = (\mathscr{C}, \mathscr{C}^*, \sim)$ is an affine plane of order $t$.*

**Theorem 7.** *For $k > t - \sqrt{t} + 1$, a $(k-1)$-cloud $\bar{\mathscr{C}}$ of index $k$ is extendable to a $k$-cloud $\bar{\mathscr{C}}$ of index $k$, so that $\bar{\Gamma}' = (\bar{\mathscr{C}}, \bar{\mathscr{C}}^*, \sim)$ is a projective plane of order $k$.*

**Proof.** If $k > t - \sqrt{t} + 1$, then $k > (t+1)/2$ and $k > t + 1 - k$. The $(k-1)$-cloud $\mathscr{C}$ defines an affine plane of order $k$. We introduce some notations, to make things easier to explain. A $\mathscr{C}\mathscr{C}^*$-line is a line intersecting $\mathscr{C}$ and $\mathscr{C}^*$. A $\mathscr{C}$-line only intersects $\mathscr{C}$, while a $\mathscr{C}^*$-line only intersects $\mathscr{C}^*$. We complete the geometry $\Gamma' = (\mathscr{C}, \mathscr{C}^*, \sim)$ with some extra elements (*special* points and lines) to a projective plane.

(i) First we show that 2 'parallel affine lines' in $\Gamma'$ define a unique (special) point. This point is not in the affine plane, but it is in the hexagon. Take two points $u_1, u_2 \in \mathscr{C}^*$, with $\Gamma_2(u_1) \cap \mathscr{C}$ and $\Gamma_2(u_2) \cap \mathscr{C}$ disjoint. We show that $\delta(u_1, u_2) = 4$ in the hexagon. Suppose $\delta(u_1, u_2) = 6$. Hence the distance between $u_2$ and a line through $u_1$ is 5. The projection of one of the $k\,\mathscr{C}\mathscr{C}^*$-lines through $u_1$ onto $u_2$, should be a $\mathscr{C}^*$-line (because 2 points of $\mathscr{C}$ are at mutual distance 4 and not 6). But as the number of $\mathscr{C}\mathscr{C}^*$-lines through a point of $\mathscr{C}^*$ (i.e., $k$) is bigger than the number of $\mathscr{C}^*$-lines through a point of $\mathscr{C}^*$ (i.e., $t+1-k$), this gives a contradiction. Hence $\delta(u_1, u_2) \neq 6$. Hence $\delta(u_1, u_2) = 4$ and $u_1 \bowtie u_2 \notin \mathscr{C}$. Put $w = u_1 \bowtie u_2$ and suppose $u_1 w$ and $u_2 w$ are $\mathscr{C}\mathscr{C}^*$-lines, with $u_i w \cap \mathscr{C} = x_i$. Then, $w = x_1 \bowtie x_2 \notin \mathscr{C}^*$, in contradiction with the definition of $\mathscr{C}^*$. Suppose $u_1 w$ is a $\mathscr{C}\mathscr{C}^*$-line, $u_1 w \cap \mathscr{C} = x_1$, and $u_2 w$ is a $\mathscr{C}^*$-line. Then the distance between $x_1$ and all points in $\Gamma_2(u_2) \cap \mathscr{C}$ is 6, again a contradiction. So $w$ is on a $\mathscr{C}^*$-line through $u_1$ and on a $\mathscr{C}^*$-line through $u_2$. All points $u_i \bowtie u_j$ obtained by this construction, will be referred to as 'special points'.

(ii) Now, we show that each parallel class defines exactly one special point. We denote this fixed parallel class by $\mathscr{C}^*_{\|}$, while the corresponding special points are in $(\mathscr{C}^*_{\|})^* = \{u_i \bowtie u_j \text{ with } u_i \neq u_j \text{ and } u_i, u_j \in \mathscr{C}^*_{\|}\}$. There are $k$ elements $u_i$ in $\mathscr{C}^*_{\|}$, each incident with $t+1-k\,\mathscr{C}^*$-lines. Each $u_i \bowtie u_j$, $u_i$ and $u_j$ distinct points in $\mathscr{C}^*_{\|}$, is on a $\mathscr{C}^*$-line, and if $u_i \bowtie u_j$ and $u_i \bowtie u_l$, with $u_i, u_j, u_l \in \mathscr{C}^*_{\|}$ and distinct, are on the same $\mathscr{C}^*$-line, the points $u_i \bowtie u_j$ and $u_i \bowtie u_l$ must coincide (as $\delta(u_j, u_l) = 4$). Also, if a special point belongs to a $\mathscr{C}^*$-line containing $u_i$, it corresponds to the parallel class of $u_i$. Hence $u_i \in \mathscr{C}^*_{\|}$ is collinear with at most $t+1-k$ elements of $(\mathscr{C}^*_{\|})^*$. Two points $u_i, u_j$ of a same parallel class are collinear with a unique special point $u_i \bowtie u_j$, and two special points are collinear with at most one $u_i$ (otherwise there arises a $k$-gon with $k < 6$). Hence the geometry $\Gamma_{\|} = (\mathscr{C}^*_{\|}, (\mathscr{C}^*_{\|})^*, \sim)$ is a linear space, with $k$ points and at most $t+1-k$ lines through a point. If there exists a triangle in $\Gamma_{\|}$, there are at most $t+1-k$ points on every line.

Now we count on different ways the pairs $(q, L)$ with $q$ a point of $\Gamma_{\|}$, $L$ a line of $\Gamma_{\|}, q \,\mathrm{I}\, L$, and $p \,\mathrm{I}\, L, p \neq q$ with $p$ fixed; further we assume the existence of a triangle in $\Gamma_{\|}$. We obtain

$$\begin{aligned} (k-1) &\leqslant (t+1-k)(t+1-k-1) \\ 0 &\leqslant k^2 - 2k - 2kt + t^2 + t + 1. \end{aligned} \qquad (*)$$

Solving for $k$, the roots of the associated equation are $k = t + 1 \pm \sqrt{t}$, or $t + 1 - k = \pm\sqrt{t}$. As we assumed $t+1-k < \sqrt{t}$ and clearly $t+1-k > -\sqrt{t}$, the quadratic form $(*)$ is negative, hence the inequality is false, so $\Gamma_{\|}$ cannot be a non-degenerate linear space. Hence $\Gamma_{\|}$ is a unique line with $k$ points on it. Translated to $\Gamma' = (\mathscr{C}, \mathscr{C}^*, \sim)$: each parallel class of affine lines defines a unique special point. The set of all special points constructed in this way, is denoted by $W$.

(iii) Subsequently we show that all points in $\mathscr{C} \cup W$ are at mutual distance 4 (this is a first step in proving that $\mathscr{C} \cup W$ is a cloud). First we look at $\delta(w, x)$, $w \in W$, $x \in \mathscr{C}$. A point $w \in W$ is at distance 2 of $k$ points $u_i$ of $\mathscr{C}^*$, belonging to the same parallel class of lines in $\Gamma'$. These lines $u_i$ cover all $k^2$ points of $\Gamma'$, hence all $k^2$ points of $\mathscr{C}$ are at distance 4 of $w$. Now we look at $\delta(w_1, w_2)$, $w_1, w_2 \in W$. There are $k\mathscr{C}^*$-lines through $w_i$, hence there are $t+1-k$ lines through $w_i$ not intersecting $\mathscr{C}^*$.

Suppose $\delta(w_1, w_2) = 6$. The projection of a $\mathscr{C}^*$-line through $w_1$ onto $w_2$ cannot be a $\mathscr{C}^*$-line through $w_2$ because points of $\mathscr{C}^*$, belonging to different parallel classes of lines in $\Gamma'$, are at distance 4. Hence the $k\mathscr{C}^*$-lines through $w_1$ should all be mapped onto (different) lines through $w_2$ but not intersecting $\mathscr{C}^*$. As there are only $t+1-k$ of these lines, this situation is impossible, hence $\delta(w_1, w_2) \neq 6$.

Clearly, $\delta(w_1, w_2) = 2$ would imply the existence of a $k$-gon with $k < 6$. Hence $\delta(w_1, w_2) = 4$, and $w_1 \bowtie w_2 \notin \mathscr{C}^*$. Also, it is easy to show that the line $N_i$ joining $w_i$ and $w_1 \bowtie w_2$ is not a $\mathscr{C}^*$-line, $i = 1, 2$. So $N_i$ is one of the $t+1-k$ lines through $w_i$ which is not a $\mathscr{C}^*$-line, $i = 1, 2$. If we put $W^* = \{w_i \bowtie w_j | w_i, w_j \in W\}$, the geometry $\Gamma_* = (W, W^*, \sim)$ is a linear space with $k+1$ points and at most $t+1-k$ lines through a point (to verify this, one can use exactly the same arguments as used in part (ii) of this proof). By (nearly) the same counting argument, one concludes that $\Gamma_*$ is degenerate, hence $W^*$ is a singleton, containing the unique point $w_* \notin \mathscr{C}^*$.

(iv) At this point, we can finish the proof: $\mathscr{C} \cup W$ is a $k$-cloud of index $k$, which means that all points of $\mathscr{C} \cup W$ are at mutual distance 4, and for $x, y \in \mathscr{C} \cup W$, $x \neq y : x \bowtie y$ is collinear with $k+1$ points of $\mathscr{C} \cup W$. Indeed, for $x, y$ both in $\mathscr{C}$, we know that $x \bowtie y$ is collinear with $k$ points of $\mathscr{C}$ and with 1 point of $W$ (the unique special point on the line $x \bowtie y$ in $\Gamma'$). For $x$ in $\mathscr{C}$ and $y$ in $W$, the point $x \bowtie y$ is in $\Gamma'$ the unique line through $x$ of the parallel class corresponding with the special point $y$. So $x \bowtie y$ is an element of $\mathscr{C}^*$, and hence collinear with $k+1$ points of $\mathscr{C} \cup W$. For $x, y$ both in $W$, we know that $x \bowtie y = w^*$, and $w^*$ is collinear with all $k+1$ points of $W$; and as there should be no ordinary quadrangles, $w^*$ cannot be collinear with any point of $\mathscr{C}$ (indeed, take $y \in \mathscr{C}$; $y$ is collinear with some point $a \in \mathscr{C}^*, a$ is collinear with a unique point $b \in W$, and $b$ is always collinear with $w^*$. If $y \sim w^*$, then there arises a quadrangle).

By putting $\bar{\mathscr{C}} = \mathscr{C} \cup W$ and $\bar{\mathscr{C}}^* = \mathscr{C}^* \cup \{w^*\}$, we constructed the desired extension of $\Gamma'$ to a projective plane. $\square$

**Corollary 8.** *A $(t-1)$-cloud $\mathscr{C}$ of index $t$ is extendable to a $t$-cloud $\bar{\mathscr{C}}$ of index $t$, so that $\bar{\Gamma}' = (\bar{\mathscr{C}}, \bar{\mathscr{C}}^*, \sim)$ is a projective plane of order $t$.*

## 4. $m$-Clouds in distance-2-regular hexagons

A subgeometry $\Gamma' = (\mathscr{P}', \mathscr{B}', \mathrm{I}')$ of a geometry $\Gamma = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ is an incidence structure such that $\mathscr{P}' \subseteq \mathscr{P}$, $\mathscr{B}' \subseteq \mathscr{B}$ and $\mathrm{I}' = \mathrm{I} \cap (\mathscr{P}' \times \mathscr{L}')$. The *trace* $p^q$ with $p, q$ opposite points of a generalized hexagon $\Gamma$, is the set of all elements at distance 2 of $p$ and distance 4 of $q$. A point $p$ is *distance*-2-*regular* if $|p^q \cap p^r| \geqslant 2$, for $q, r$ opposite $p$, implies $p^q = p^r$. A generalized hexagon is point-distance-2-regular if all points are distance-2-regular.

For point-distance-2-regular hexagons, $m$-clouds turn out to be well studied objects in projective planes. Such a plane is derivable from a generalized hexagon with a distance-2-regular point as follows. If $p$ is distance-2-regular and $q$ is opposite $p$, then there exists a unique weak ideal (i.e. of order $(1,t)$) subhexagon $\Gamma(p,q)$ through $p$ and $q$. If we define $\Gamma^+(p,q)$ to be the set of all points of $\Gamma(p,q)$ at distance 0 or 4 of $p$, and $\Gamma^-(p,q)$ to be the complementary pointset in $\Gamma(p,q)$, then $\Gamma_\pi = (\Gamma^+(p,q), \Gamma^-(p,q), \sim)$ is a projective plane. (See [8, Lemma 1.9.10]). If all points of $\Gamma$ are distance-2-regular, then $\Gamma$ is classical (see [4]), and every associated projective plane $\Gamma_\pi$ will be classical too (this means Desarguesian).

**Theorem 9.** *Let $\Gamma$ be a generalized hexagon of order $(s,t)$, such that all points are distance-2-regular. Let $\mathscr{C}$ be an $m$-cloud of $\Gamma$, with $x_1, x_2, x_3 \in \mathscr{C}$ and $x_1 \bowtie x_2 \neq x_1 \bowtie x_3$. The geometry $\Gamma_{\mathscr{C}} = (\mathscr{C}, \mathscr{C}^*, \sim)$ is a subgeometry of the projective plane $\Gamma_\pi = (\Gamma^+(x_3, x_1 \bowtie x_2), \Gamma^-(x_3, x_1 \bowtie x_2), \sim)$ of order $t$, such that all lines of $\Gamma_\pi$ intersect $\Gamma_{\mathscr{C}}$ in 0, 1 or $m+1$ points. The constant $f+1$ is the number of $(m+1)$-secants of $\Gamma_{\mathscr{C}}$ through a point of $\Gamma_{\mathscr{C}}$.*

**Proof.** Take the unique weak ideal subhexagon $\Gamma' := \Gamma(x_3, x_1 \bowtie x_2)$. This geometry contains the ordinary hexagon with vertices $\{x_1, x_1 \bowtie x_2, x_2, x_2 \bowtie x_3, x_3, x_3 \bowtie x_1\}$. We put $y := x_1 \bowtie x_2$. Now take a point $x_4 \in \mathscr{C}$ and suppose $x_4$ is not contained in $\Gamma'$. If $x_4 \bowtie x_i$ (for $i \in \{1,2,3\}$) is different from $x_1 \bowtie x_2, x_2 \bowtie x_3, x_3 \bowtie x_1$, the unique shortest path between $x_4$ and $x_3$ is denoted by $(x_4, M, z, L, x_3)$. As $\Gamma'$ is ideal, each line of $\Gamma$ through a point of $\Gamma$ is also a line of $\Gamma'$. So if $z$ belongs to $\Gamma', x_4 = \mathrm{proj}_M\, x_1$ also belongs to $\Gamma'$—a contradiction. Hence, $u := \mathrm{proj}_L\, y$ is different from $z$. As $x_1, x_2 \in y^{x3} \cap y^{x4}$, $y \bowtie u \in y^{x3}$, and $y$ is distance-2-regutar, $y \bowtie u$ should be in $y^{x4}$. Hence $\delta(x_4, y \bowtie u) = 4$, and there arises a pentagon through $y \bowtie u$, $u, z$ and $x_4$. This is a contradiction.

If on the other hand $x_4 \bowtie x_1$ is equal to $x_1 \bowtie x_2$ (or some equivalent condition), we put $L = \mathrm{proj}_{x_1 \bowtie x_2}\, x_4$. As $x_4 = \mathrm{proj}_L\, x_3, x_4$ belongs to $\Gamma'$, again a contradiction. Hence each point of $\mathscr{C}$ belongs to $\Gamma'$. Next, let $y_1 \in \mathscr{C}^*$, $y_1 \neq y$. Then $y_1 = x_5 \bowtie x_6$ for points $x_5, x_6 \in \mathscr{C}$. As $x_5, x_6$ are points of $\Gamma'$, also $x_5 \bowtie x_6 = y$ belongs to $\Gamma'$. So each point of $\mathscr{C}^*$ belongs to $\Gamma'$.

This shows that all points of $\mathscr{C}$ are in $\Gamma^+(x_3, x_1 \bowtie x_2)$, and all points of $\mathscr{C}^*$ are in $\Gamma^-(x_3, x_1 \bowtie x_2)$. In particular any two distinct points of $\mathscr{C}^*$ are at mutual distance 4. If a line of $\Gamma_\pi$ belongs to $\mathscr{C}^*$, it will be incident with $m+1$ points of $\Gamma_{\mathscr{C}}$. If a line does not belong to $\mathscr{C}^*$, it can (by definition of $\mathscr{C}^*$) only be incident with 0 or 1 point of $\Gamma_{\mathscr{C}}$. Clearly $f+1$ is the number of $(m+1)$-secants of $\Gamma_{\mathscr{C}}$ through a point of $\Gamma_{\mathscr{C}}$. $\square$

**Theorem 10.** *Let $\Gamma$ be a generalized hexagon of order $(s,t)$ with a distance-2-regular point $p$. Let $q$ be a point opposite $p$ and suppose $\mathscr{C}$ is a subset of the pointset of the projective plane $\Gamma_\pi = (\Gamma^+(p,q), \Gamma^-(p,q), \sim)$, such that all lines of $\Gamma_\pi$ intersect $\mathscr{C}$ in $0, 1$ or $m+1$ points. Then $\mathscr{C}$ is an $m$-cloud of $\Gamma$.*

**Proof.** Immediate. $\square$

### *4.1. Examples*

Let $\Gamma$ be a generalized hexagon of order $(s,t)$, with a distance-2-regular point $p$ and $\Gamma_\pi$ as above. A conic in $\Gamma_\pi$ corresponds with a 1-cloud of index $t-1$ of $\Gamma$. A maximal arc of type $(0,m)$ in $\Gamma_\pi$ corresponds with an $(m-1)$-cloud of index $t$ of $\Gamma$. Unitals in $\Gamma_\pi$ correspond with $\sqrt{t}$-clouds of index $t-1$ of $\Gamma$. Baersubplanes in $\Gamma_\pi$ correspond to $\sqrt{t}$-clouds of index $\sqrt{t}$ of $\Gamma$.
Baersubplanes are special subplanes of a given plane. But any subplane of $\Gamma_\pi$ corresponds with a certain cloud, as stated in the following corollary.

**Corollary 11.** *For $\Gamma$ a point-distance-2-regular hexagon of order $(s, p^h)$, there exists a $p^i$-cloud of index $p^i$ for every $i$ dividing $h$, as well as a $(p^i-1)$-cloud of index $p^i$.*

If we focus on very small subplanes of a given plane, we have a result about sets of 4 points $x_i$ at mutual distance 4, such that all $x_i \bowtie x_j$ are different. Such a set is a 1-cloud of index 2, and corresponds with the affine plane of order 2, contained in every projective plane—unlike the projective plane of order 2.

**Corollary 12.** *Let $\Gamma$ be a generalized hexagon of order $(s,t)$, such that all points are distance-2-regular, and $t$ odd. Then a 1-cloud of index 2 in $\Gamma$ is not extendable to a 2-cloud of index 2.*

**Proof.** If the converse were true, the Fano-plane PG(2,2) would be contained in a classical projective plane of odd order. □

## 5. $m$-Clouds in anti- regular hexagons

Let $\Gamma$ be a generalized hexagon with 3 distinct points $p,u,v$ such that $\delta(p,u)=6=\delta(p,v)$. We introduce the following subset of the intersection of the traces $p^u$ and $p^v$:

$$p^{\{u,v\}}=\{x\in p^u\cap p^v \mid \mathrm{proj}_x u\neq \mathrm{proj}_x v\}.$$

A generalized hexagon of order $q$ is *anti-regular* if $|p^{\{u,v\}}|\geqslant 2$ implies $|p^u\cap p^v|=3$ and $|p^{\{u,v\}}|=3$ for all traces $p^u$, $p^v$. A finite generalized hexagon $\Gamma$ of order $q$ is anti-regular if and only if $\Gamma$ is isomorphic to the dual Split–Cayley hexagon $H(q)^{\mathrm{D}}$ with $q$ not divisible by 3. (This characterization can be found in [1].)

**Theorem 13.** *Suppose $\Gamma$ is a generalized hexagon of order $q$. If $\Gamma$ is anti-regular, then $\Gamma$ contains no $m$-cloud for $m\geqslant 2$ with $|\mathscr{C}*|>1$.*

**Proof.** Take a point $p\in\mathscr{C}^*$ collinear with $x,y,z\in\mathscr{C}$. Let $u\in\mathscr{C}$ be at distance 6 of $p$. Consider $u\bowtie z\in\mathscr{C}^*$. This point is collinear with a third point of $\mathscr{C}$, say $v$. Put $L=\mathrm{proj}_v x$ and $M=\mathrm{proj}_v y$ As there are no pentagons in $\Gamma$, $\mathrm{proj}_x v\neq\mathrm{proj}_x u$ and $L\neq M$. But now we have $x,y,z\in p^v\cap p^u$ with $\mathrm{proj}_x u\neq\mathrm{proj}_x v, \mathrm{proj}_y u\neq\mathrm{proj}_y v$ and $\mathrm{proj}_z u=\mathrm{proj}_z v$. This is in contradiction with the antiregularity of $\Gamma$. □

## 6. Remark

As the existence of $(t-1)$-clouds of index $(t-1)$ in point-distance-2-regular generalized hexagons is impossible, we could wonder whether such a cloud can exist in a non-classical generalized hexagon. We tried the extended Higman–Sims technique (see [3, p. 9; 2, p. 144]) for proving the non-existence of those clouds in non-classical generalized hexagons, but unfortunately, this gives no usable result.

## 7. $m$-Clouds in generalized quadrangles

As for generalized hexagons, we can define an $m$-cloud $\mathscr{C}$ of a generalized quadrangle to be a set of points at mutual distance 4, such that $\forall x,y\in\mathscr{C}$: $\Gamma_2(x)\cap\Gamma_2(y)$ is collinear with exactly $m+1$ points of $\mathscr{C}$. But as quadrangles are now allowed, one cannot compute the size of $\mathscr{C}$ as done in Theorem 1. So we could define a *proper m-cloud* to be an $m$-cloud such that no 4 points of $\mathscr{C}\cup\mathscr{C}^*$ form an ordinary quadrangle. In this way, counting is possible, but this is still not sufficient for deriving good results from the extended Higman–Sims technique—whereas this technique is very useful in the case of the most degenerate $m$-cloud possible: if $\forall x, y \in \mathscr{C}$, $\forall u, v \in \mathscr{C}^*$: $x, y, u, v$ form a quadrangle, then $|\mathscr{C}| = m+1$, $|\mathscr{C}^*| = n+1$ and $(m+1)(n+1) \leqslant s^2$. (See [3, p. 11]). However, by computer-search, we can tell something about the smallest possible proper $m$-cloud of index $m$ in some classical quadrangles of odd order. This cloud is a 2-cloud of index 2, and is in fact the double of a Fano-plane. Let $Q(5,s)$ (resp. $Q(4,s)$) be the generalized quadrangle of order $(s,s^2)$ (resp. $(s,s)$) consisting of all points and lines on the elliptic quadric in PG$(5,s)$ (resp. parabolic quadric in PG$(4,s)$). Then we showed that $Q(5,3)$ and $Q(4,5)$ do not contain 2-clouds of index 2, whereas $Q(4,7)$, $Q(4,11)$ and $Q(4,13)$ do contain 2-clouds.

### 7.1. From m-clouds to dense clouds

For generalized quadrangles, a derived notion is that of a *dense cloud*. It is inspired by taking $\mathscr{C}$ and $\mathscr{C}^*$ together in one set $\mathscr{D}$. A dense cloud $\mathscr{D}$ of index $a$ is a set of $d$ points such that any point $p$ of $\mathscr{D}$ is collinear with exactly $a$ points of $\mathscr{D}\backslash\{p\}$. Then, with the Higman–Sims technique, we can prove that $d \leqslant (a+t+1)(st+1)/(t+1)$. If $d$ attains this bound, then every point outside $\mathscr{D}$ is collinear with exactly $a+t+1$ points of $\mathscr{D}$, and $\mathscr{D}$ is called maximal.

**Remark.** We have also $d \geqslant (s+1)(a+1-s)$ with equality if and only if every point outside $\mathscr{D}$ is collinear with exactly $a+1-s$ points of $\mathscr{D}$ (see 1.10.1 of [3]).

**Theorem 14.** *Let $\Gamma$ be a generalized quadrangle, and let $\mathscr{D}$ be a dense cloud of index $a$ of $\Gamma$. If $|\mathscr{D}| = (a+t+1)(st+1)/(t+1)$, then every line of $\Gamma$ is incident with a constant number of points of $\mathscr{D}$, this constant being equal to $a/(t+1)+1$.*

**Proof.** Take a line $L$ of $\Gamma$ and suppose $L$ intersects $\mathscr{D}$ in $k$ points. Each point of $\mathscr{D}$ on $L$ is collinear with $a-k+1$ other points of $\mathscr{D}$, and as $|\mathscr{D}|$ attains the bound $((a+t+1)(st+1))/(t+1)$, each point off $\mathscr{D}$ on $L$ is collinear with $(a+t+l)-k$ points of $\mathscr{D}$ not on $L$. As all points of $\Gamma$ are at distance at most 3 of $L$, we counted all points of $\mathscr{D}$ in this way. Hence $k+k(a-k+1)+(s+1-k)(a+t+1-k) = |\mathscr{D}|$, implying that $k$ is equal to $a/(t+1)+1$. $\square$

**Corollary 15.** *With notations as above and with the terminology of* [7], *the maximal dense clouds of a generalized quadrangle $\Gamma$ of order $(s,t)$ are the $(a/(t+1)+1)$-ovoids of $\Gamma$.*

The generalized quadrangle $Q(5,q)$ of order $(q,q^2)$ is the dual of the hermitian polar space $H(3,q^2)$ in 3 dimensions. Segre [5] shows that, if there is a subset $K$ of the lineset of $H(3,q^2)$, such that through every point of $H(3,q^2)$ there pass exactly $m$ lines of $K$, this set $K$ is either the set of all lines of $H(3,q^2)$ or $m = (q+1)/2$. If $m = (q+1)/2$, such a set of lines is called a hemisystem of $H(3,q^2)$. By dualizing this, we obtain the following: the proper maximal dense clouds of the generalized quadrangle $Q(5,q)$ are the $(q+1)/2$-ovoids. At present such a $(q+1)/2$-ovoids is only known for $q=3$; it is the 56-cap of Hill in PG(5,3).

### 7.2. Examples

Let $\Gamma$ be a generalized quadrangle of order $(s,t)$. The pointset of each subquadrangle of order $(s',t')$ is a non-maximal dense cloud of index $s'(t'+1)$. The set $\Gamma_2(x)$ of all points at distance 2 of a given point $x$ is a dense cloud of index $s-1$, but is never maximal. Each partial ovoid of $\Gamma$ is a dense cloud of index 0, while each ovoid of $\Gamma$ is a maximal dense cloud of index 0. Each union of $1+i$ disjoint ovoids is a maximal dense cloud of index $i(t+1)$.

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